Objectives

- Data structure: Heaps
- Data structure: Graphs

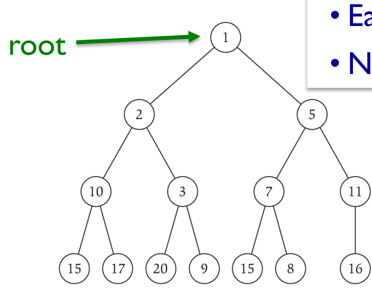
Submit problem set 2

Review

- What is a priority queue?
- What is a heap?
  - Properties
  - Implementation
- What is the process for finding the smallest element in a heap?
- What is the process for adding to a heap?
  - What is the runtime of adding to a heap?

Review: Heap Defined

- Combines benefits of sorted array and list
- Balanced binary tree

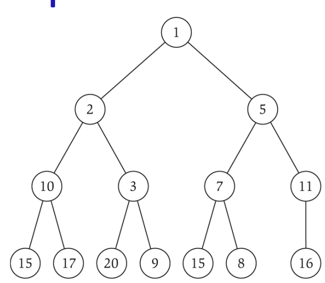


- Each node has *at most* 2 children
- Node value is its *key*

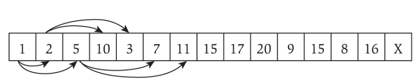
Heap order: each node's key is at least as large as its parent's

Note: **not** a binary search tree

Review: Implementing a Heap



- Option 1: Use pointers
  - Each node keeps
    - Element it stores (key)
    - 3 pointers: 2 children, parent
- Option 2: No pointers
  - Requires knowing upper bound on *n*
  - For node at position *i*
    - left child is at *2i*
    - right child is at *2i+1*



Review: Implementing a Heap

- Finding the minimal element
  - First element
  - O(1)

Review: Heapify-Up

Heap

Position where node added

```
Heapify-up(H, i):
   if i > 1 then
       j=parent(i)=floor(i/2)
       if key[H[i]] < key[H[j]] then
          swap array entries H[i] and H[j]
          Heapify-up(H, j)
```

Heapify-Up

- Claim. Assuming array H is almost a heap with key of `H[i]` too small, `Heapify-Up` fixes the heap property in O(log i) time
  - Can insert a new element in a heap of *n* elements in O(log n) time
- Proof. By induction
  - If i=1 …

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- Proof. By induction
  - If i=1, is already a heap → O(1)
  - If i>1, …

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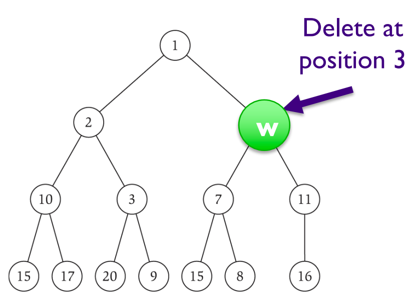
  - Can insert a new element in a heap of *n* elements in O(log n) time
- Proof. By induction
  - If i=1, is already a heap → O(1)
  - If i>1,
    - Swaps are O(1)
    - Swaps continue up to root (max) → log i

Deleting an Element



Deleting an Element

- Delete at position *i*
- Removing an element:
  - Messes up heap order
  - Leaves a "hole" in the heap
- Not as straightforward as `Heapify-Up`
- Algorithm:
  1. Fill in element where hole was
     - Patch hole: move n^{th} element into i^{th} spot
  2. Adjust heap to be in order
     - At position *i* because moved n^{th} item up to *i*

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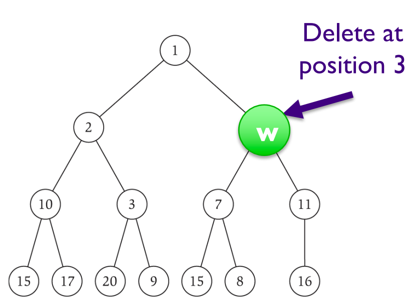
Deleting an Element



Example of OK:
11 deleted, replaced by 16

- Two "bad" possibilities: element *w* is
  - Too small: violation is between it and parent → `Heapify-Up`
  - Too big: with one or both children → `Heapify-Down` (example: w becomes 12)

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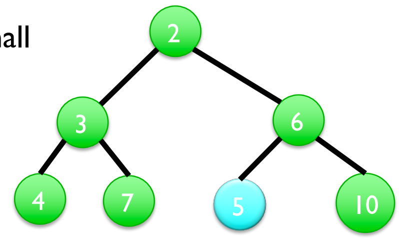
Deleting an Element

Example where new key is too small



- Delete 9
- Replace with 5 (from other side of heap)
- But 5 < 6, so need to `Heapify-Up`

Heapify-Down

```
Heapify-down(H, i):
   n = length(H)
   if 2i > n then            Why can we stop?
      Terminate with H unchanged
   else if 2i < n then
      left=2i and right=2i+1
      j be index that minimizes
            key[H[left]] and key[[H[right]]
   else if 2i = n then
      j=2i

   if key[H[j]] < key[H[i]] then
      swap array entries H[i] and H[j]
      Heapify-down(H, j)
```

Heapify-Down

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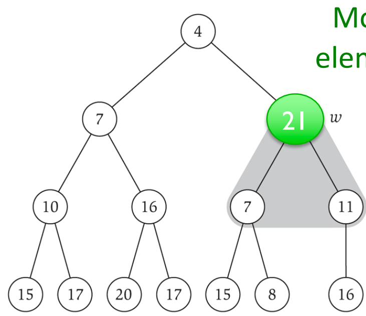
```

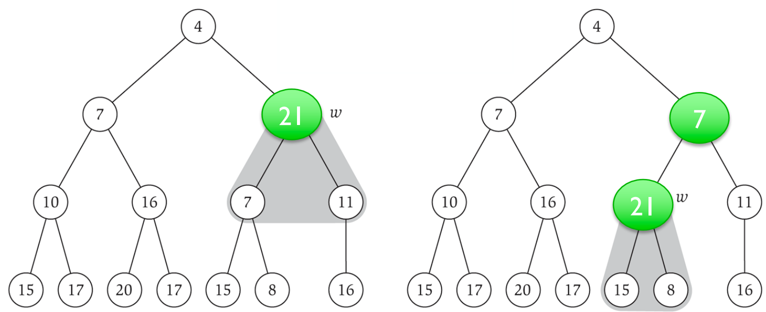
i is a leaf – nowhere to go

Practice: Heapify-Down

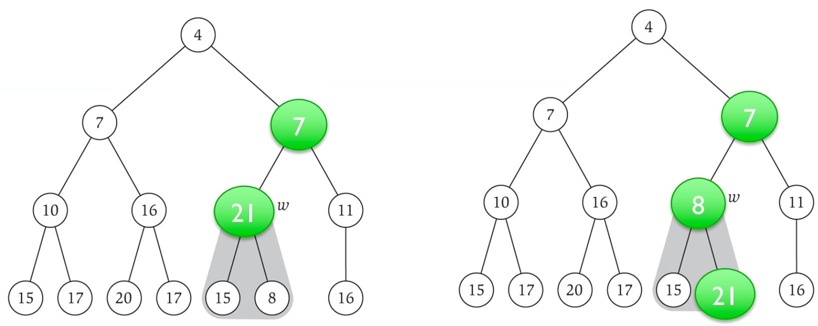


Moved 21 to where element was removed

Practice: Heapify-Down



Practice: Heapify-Down



Runtime of Heapify-Down?

```
Heapify-down(H, i):
   n = length(H)
   if 2i > n then
      Terminate with H unchanged
   else if 2i < n then
      left=2i and right=2i+1
      j be index that minimizes   O(1)
            key[H[left]] and key[[H[right]]
   else if 2i = n then
      j=2i

   if key[H[j]] < key[H[i]] then
      swap array entries H[i] and H[j] O(1)
      Heapify-down(H, j)
```

Num swaps: O(log n)

Implementing Priority Queues with Heaps

Operation	Description	Run Time
StartHeap(N)	Creates an empty heap that can hold N elements	
Insert(v)	Inserts item v into heap	
FindMin()	Identifies minimum element in heap but does not remove it	
Delete(i)	Deletes element in heap at position i	
ExtractMin()	Identifies and deletes an element with minimum key from heap	

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Comparing Data Structures

Operation	Heap	Unsorted List	Sorted List
Start(N)		O(1)	O(1)
Insert(v)		O(1)	O(n)
FindMin()		O(1)	O(1)
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ExtractMin()		O(n)	O(1)

Comparing Data Structures

Operation	Heap	Unsorted List	Sorted List
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Delete(i)	O(log n)	O(n)	O(1)
ExtractMin()	O(log n)	O(n)	O(1)

Putting It All Together…

1. Add elements into PQ with the number's value as its priority
2. Then extract the smallest number until done
   - Come out in sorted order

What is the running time of sorting numbers using a PQ implemented with a **heap**?

O(n log n)

Additional Heap Operations

- Access elements in PQ by "name"

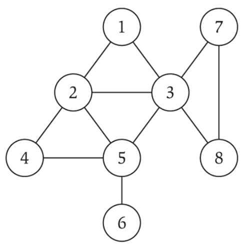
Key	2	4	5	6	9	20	Priority
Value	3542	5143	8712	1264	9123	5954	Process id

  - Maintain additional array `Position` that stores current position of each element in heap
- Operations:
  - Delete(Position[v])
    - Does not increase overall running time
  - ChangeKey(v, α)
    - Changes key of element v to α
    - Identify position of element v in array (`Position` array)
    - Change key, heapify

GRAPHS

Undirected Graphs G = (V, E)

- V = nodes (vertices)
- E = edges between pairs of nodes
- Captures pairwise relationship between objects
- Graph size parameters: n = |V|, m = |E|

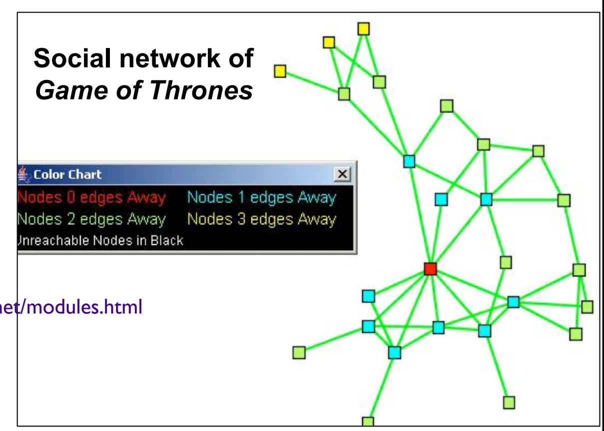


V = { 1, 2, 3, 4, 5, 6, 7, 8 }
E = { 1-2, 1-3, 2-3, 2-4, 2-5, 3-5, 3-7, 3-8, 4-5, 5-6 }
n = 8
m = 11

Social Networks

- Node: people; Edge: relationship between 2 people
- *Everything Bad Is Good for You: How Today's Popular Culture Is Actually Making Us Smarter*

Television shows have complex plots, complex social networks



http://www.cs.duke.edu/csed/harambeenet/modules.html

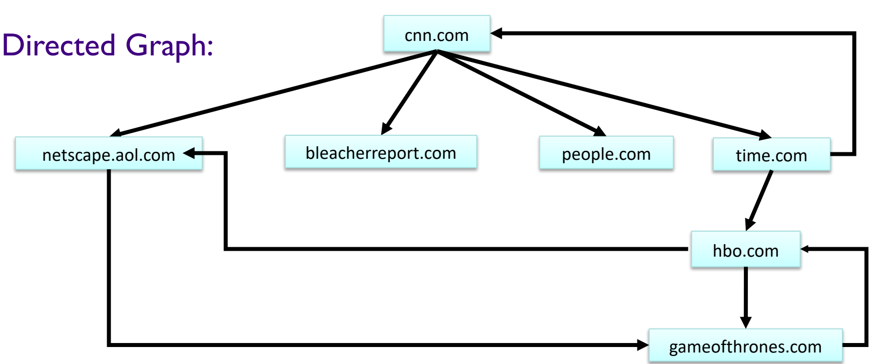
Facebook: Visualizing Friends

http://www.facebook.com/notes/facebook-engineering/visualizing-friendships/469716398919

World Wide Web

- Web graph
  - Node: web page
  - Edge: hyperlink from one page to another

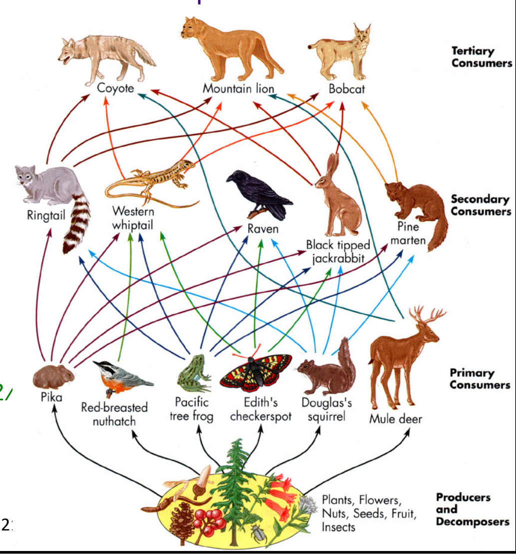
Directed Graph:



Ecological Food Web

- Food web graph
  - Node = species
  - Edge = from prey to predator

Directed Graph:



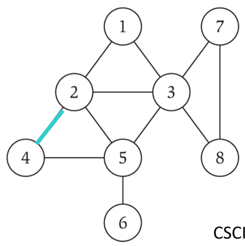
Reference:
`https://www.msu.edu/course/isb/202/rtmay/images/foodweb.jpg`

Graph Applications

Graph	Nodes	Edges
transportation	street intersections	highways
communication	computers	fiber optic cables
World Wide Web	web pages	hyperlinks
social	people	relationships
food web	species	predator-prey
software systems	functions	function calls
scheduling	tasks	precedence constraints
circuits	gates	wires

Graph Representation: Adjacency Matrix

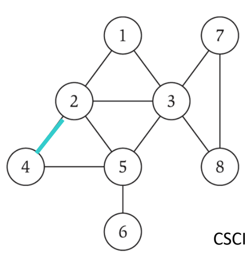
- n × n matrix with $A_{uv} = 1$ if (u, v) is an edge
  - Two representations of each edge (symmetric matrix)
  - Space?
  - Checking if (u, v) is an edge?
  - Identifying all edges?



	1	2	3	4	5	6	7	8
1	0	1	1	0	0	0	0	0
2	1	0	1	1	1	0	0	0
3	1	1	0	0	1	0	1	1
4	0	1	0	0	1	0	0	0
5	0	1	1	1	0	1	0	0
6	0	0	0	0	1	0	0	0
7	0	0	1	0	0	0	0	1
8	0	0	1	0	0	0	1	0

Graph Representation: Adjacency Matrix

- n × n matrix with $A_{uv} = 1$ if (u, v) is an edge
  - Two representations of each edge (symmetric matrix)
  - Space: $\Theta(n^2)$
  - Checking if (u, v) is an edge: $\Theta(1)$ time
  - Identifying all edges: $\Theta(n^2)$ time

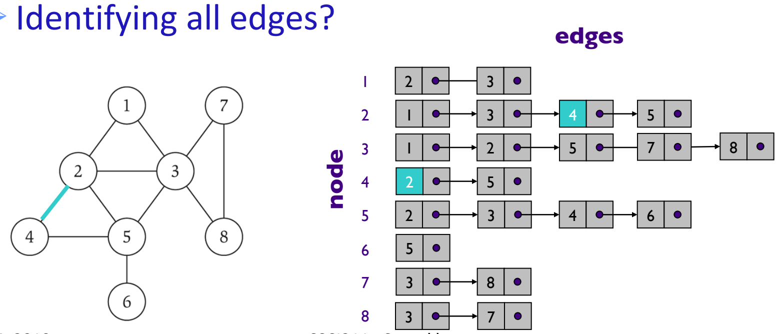


	1	2	3	4	5	6	7	8
1	0	1	1	0	0	0	0	0
2	1	0	1	1	1	0	0	0
3	1	1	0	0	1	0	1	1
4	0	1	0	0	1	0	0	0
5	0	1	1	1	0	1	0	0
6	0	0	0	0	1	0	0	0
7	0	0	1	0	0	0	0	1
8	0	0	1	0	0	0	1	0

Graph Representation: Adjacency List

- Node indexed array of lists
  - Two representations of each edge
  - Space?
  - Checking if (u, v) is an edge?
  - Identifying all edges?

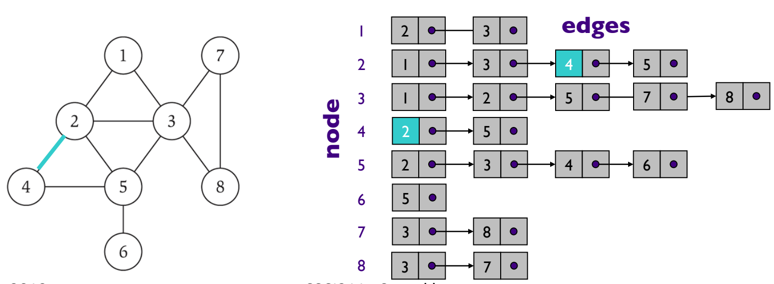
What are the extremes?



Graph Representation: Adjacency List

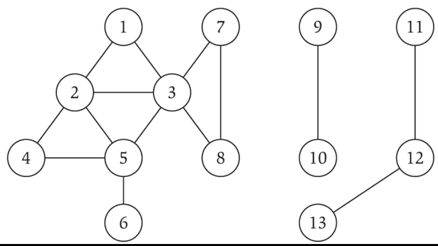
- Node indexed array of lists
  - Two representations of each edge
  - Space = 2m + n = O(m + n)
  - Checking if (u, v) is an edge takes O(deg(u)) time
  - Identifying all edges takes Θ(m + n) time

degree = number of neighbors of u



Paths and Connectivity

- Def. A ***path*** in an undirected graph G = (V, E) is a sequence P of nodes $v_1, v_2, \ldots, v_{k-1}, v_k$
  - Each consecutive pair v_i, v_{i+1} is joined by an edge in E
- Def. A path is ***simple*** if all nodes are *distinct*
- Def. An undirected graph is ***connected*** if ∀ pair of nodes u and v, there is a path between u and v



- *Short path*
- *Distance*